International Journal of Engineering Research & Management Technology
ISSN: 2348-4039
November-December-2019 Volume 6, Issue-6
Email: editor@ijermt.org www.ijermt.org

GENERATING FUNCTIONS FOR LAGUERRE AND LEGENDRE GEGENBAUER POLYNOMIALS

Deepshikha Sharma
Research Scholar
Dr Sunita Pannu
Supervisor
Calorx Teacher's University

Abstract

The Laguerre, Legendre, and Gegenbauer polynomials are interconnected through various relationships and transformations. For example, the Legendre polynomials can be expressed as a special case of the Gegenbauer polynomials. Additionally, there exist connections between these polynomials and other special functions, such as the Hermite polynomials and the Bessel functions. The applications of these polynomials extend beyond their individual domains. They are used in solving boundary value problems, approximating functions, and analyzing physical phenomena. Legendre polynomials are fundamental in the theory of spherical harmonics, which are essential in describing physical phenomena on the sphere, such as the Earth's magnetic field and the propagation of electromagnetic waves. They also play a role in numerical analysis and approximation theory. Laguerre polynomials have been utilized in a wide variety of other contexts, such as the Blissard problem, the representation of Lucas polynomials of the first and second kinds, the recurrence relations for a class of Freud-type polynomials, the representation of symmetric functions of a countable set of numbers, and furthermore, the generalization of the classical algebraic Newton-Girard formulas.

Keywords:

Laguerre, Legendre, Gegenbauer, Polynomials, Functions

Introduction

According to the following theorem, the new generating relation can be derived from the bilateral function that has been provided.

Theorem 1.1.1 If there exists a generating function of the form

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n w^n L_n^{(\alpha)}(x)P_m^{(n,\beta)}(u), \tag{1.1.1}$$

then

$$\begin{aligned}&\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha}G\left(x(1+w),\tfrac{u+wt}{1-wt},\tfrac{w}{1-wt}\right)\\ &=\sum_{n,p,q=0}^{\infty}a_n w^{n+p+q}\frac{(1+n)_p(1+n+\alpha+m)_q}{p!q!}L_{(n+p)}^{(\alpha-p)}(x)P_m^{(n+q,\beta)}(u)t^q.\end{aligned} \tag{1.1.2}$$

Proof: Moving on, let us proceed with the linear partial differential operators that are listed below.

$$R_1=xy^{-1}z\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-xy^{-1}z, \tag{1.1.3}$$

and

$$R_2=(1+u)t\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(1+\beta+m)t. \tag{1.1.4}$$

So that

$$R_1[y^{\alpha}z^nL_n^{(\alpha)}(x)]=(1+n)L_{(n+1)}^{(\alpha-1)}(x)y^{(\alpha-1)}z^{(n+1)}, \tag{1.1.5}$$

and

$$R_2[t^nP_m^{(n,\beta)}(u)]=(1+n+\beta+m)P_m^{(n+1,\beta)}(u)t^{(n+1)}. \tag{1.1.6}$$

Also, we have

$$\exp(wR_1)f(x,y,z) = \exp\left(\frac{-wxz}{y}\right)f(x+wxy^{-1}z, y+wz, z), \quad (1.1.7)$$

and

$$\exp(wR_2)f(u,t) = (1-wt)^{-(1+\beta+m)}f\left(\frac{u+wt}{1-wt}, \frac{t}{1-wt}\right). \quad (1.1.8)$$

Next, we will consider the generating function (1.1.1) and replace the w in it with wtz. After that, we will multiply both sides by y^{α}, which will result in the following:

$$y^{\alpha}G(x,u,wtz) = y^{\alpha}\sum_{n=0}^{\infty} a_n (wtz)^n L_n^{(\alpha)}(x) P_m^{(n,\,\beta)}(u). \quad (1.1.9)$$

Using the functions exp(wR1) and exp(wR2) on both sides of the equation (1.1.9), we have

$$\exp(wR_1)\exp(wR_2)\left[y^{\alpha}G(x,u,wtz)\right]$$
$$= \exp(wR_1)\exp(wR_2)\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) y^{\alpha} P_m^{(n,\beta)}(u)(wtz)^n. \quad (1.1.10)$$

The left-hand side of the equation (1.1.10) can be simplified with the assistance of the equations (1.1.7) and (1.1.8).

$$\exp\left(\frac{-wxz}{y}\right)(1-wt)^{-(1+\beta+m)}(y+wz)^{\alpha}G\left(x+wxy^{-1}z, \tfrac{u+wt}{1-wt}, \tfrac{wtz}{1-wt}\right). \quad (1.1.11)$$

As an additional point of interest, the right-hand side of (1.1.10) is simplified with the assistance of (1.1.5) and (1.1.6).

$$\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p}{p!}L_{n+p}^{(\alpha-p)}(x)y^{\alpha-p}\frac{(1+n+\beta+m)_q}{q!}$$
$$\times P_m^{(n+q,\,\beta)}(u)z^{n+p}t^{n+q}. \quad (1.1.12)$$

Due to this, the simplified form of the expression (1.1.10) is

$$\exp\left(\frac{-wxz}{y}\right)(1-wt)^{-(1+\beta+m)}(y+wz)^{\alpha}G\left(x+wxy^{-1}z,\frac{u+wt}{1-wt},\frac{wtz}{1-wt}\right)$$
$$=\sum_{n,p,q=0}^{\infty}a_n w^{n+p+q}\frac{(1+n)_p(1+n+\beta+m)_q}{p!q!}L_{n+p}^{(\alpha-p)}(x)P_m^{(n+q,\beta)}(u)$$
$$\times y^{\alpha-p}z^{n+p}t^{n+q}. \quad (1.1.13)$$

A bidirectional generating function (1.1.14) for generalized modified Laguerre and Jacobi polynomials is obtained by finally inserting z/y = 1 in the equation (1.1.13).

$$\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha}G\left(x+wx,\frac{u+wt}{1-wt},\frac{w}{1-wt}\right)=$$
$$\sum_{n,p,q=0}^{\infty}a_n w^{n+p+q}\frac{(1+n)_p(1+n+\beta+m)_q}{p!q!}L_{n+p}^{(\alpha-p)}(x)P_m^{(n+q,\beta)}(u)t^q. \quad (1.1.14)$$

Finally, the proof of the theorem is finished with this.

Theorem 1.1.2 In the event that there is a bilateral producing relation known as the form

$$G(x,v,w)=\sum_{n=0}^{\infty}a_n w^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v), \quad (1.1.15)$$

then

$$\left(\frac{1+w}{1+2w}\right)^{\alpha}\exp(-wv)\ G\left(\frac{x+2w}{1+2w},v+wv,w\right)$$
$$=\sum_{n,p,q=0}^{\infty}a_n w^{n+q}\frac{(1+n)_q}{q!}P_{n+p}^{(\alpha,\beta-p)}(x)L_{(n+q)}^{(\alpha-q)}(v). \quad (1.1.16)$$

Proof The variables x, y, and z in the operator R_1 are exchanged for the variables v, s, and t, respectively, at this point. The operator R_1 can be rewritten as follows with the help of this replacement:

$$R_1=vs^{-1}t\frac{\partial}{\partial v}+t\frac{\partial}{\partial s}-vs^{-1}t.$$

So that

$$R_1\left(s^\alpha t^n L_n^{(\alpha)}(v)\right) = (1+n)L_{(n+1)}^{(\alpha-1)}(v)s^{(\alpha-1)}t^{(n+1)}. \tag{1.1.17}$$

Let us begin by defining the R_3 operator.

$$\begin{aligned} R_3 &= (1-x^2)y^{-1}z\frac{\partial}{\partial x} - z(x-1)\frac{\partial}{\partial y} - (1+x)y^{-1}z^2\frac{\partial}{\partial z} \\ &- (1+\alpha)(1+x)y^{-1}z. \end{aligned} \tag{1.1.18}$$

Operating R3 on $y^\beta z^n P^{(\alpha,\beta)}{}_n(x)$, we get

$$R_3\left(y^\beta z^n P_n^{(\alpha,\beta)}(x)\right) = -2(1+n)P_{n+1}^{(\alpha,\beta-1)}(x)y^{\beta-1}z^{n+1}. \tag{1.1.19}$$

Also, we have

$$\exp(wR_3)f(x,y,z) = \left(\frac{y}{y+2wz}\right)^{\alpha+1} f\left(\frac{xy+2wz}{y+2wz}, \frac{y(y+2wz)}{y+2wz}, \frac{yz}{y+2wz}\right), \tag{1.1.20}$$

and

$$\exp(wR_1)f(v,s,t) = \exp\left(\frac{-wvt}{s}\right)f(v+wvs^{-1}t, s+wt, t). \tag{1.1.21}$$

Now, we consider (1.1.15) and replacing there w by wtz; and then multiplying both sides by $y^\beta s^\alpha$, we get

$$y^\beta s^\alpha G(x,v,wtz) = y^\beta s^\alpha \sum_{n=0}^{\infty} a_n (wtz)^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v). \tag{1.1.22}$$

Operating $\exp(wR_1)$, $\exp(wR_3)$ on both sides of (1.1.22), we have

$$\exp(wR_1)\exp(wR_3)\left[y^{\beta}s^{\alpha}G(x,v,wtz)\right]$$
$$= \exp(wR_1)\exp(wR_3)\sum_{n=0}^{\infty} a_n(wtz)^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v)y^{\beta}s^{\alpha}. \quad (1.1.23)$$

With the help of (1.1.17) and (1.1.19) the right hand side of (1.1.23) can be simplified as

$$\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p}{p!}\frac{(1+n)_q}{q!}(-2)^p P_{n+p}^{(\alpha,\beta-p)}(x)L_{n+q}^{(\alpha-q)}(v)$$
$$\times y^{(\beta-p)}s^{(\alpha-q)}z^{(n+p)}t^{(n+q)}. \quad (1.1.24)$$

Also, the left hand side of (1.1.23) with the help of (1.1.20) and (1.1.21) is simplified as

$$y^{\beta}(s+wt)^{\alpha}\exp\left(\tfrac{-wvt}{s}\right)\left(\tfrac{y}{y+2wz}\right)^{\alpha+1}G\left(\tfrac{xy+2wz}{y+2wz}, v+wvs^{-1}t, \tfrac{wtyz}{y+2wz}\right). \quad (1.1.25)$$

Therefore, the simplified form of (1.1.23) is

$$y^{(\alpha+\beta+1)}\left(\tfrac{s+wt}{y+2wz}\right)^{\alpha}\exp\left(\tfrac{-wvt}{s}\right)(y+2wz)^{-1}G\left(\tfrac{xy+2wz}{y+2wz}, v+wvs^{-1}t, \tfrac{wtyz}{y+2wz}\right)$$
$$= \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p}{p!}\frac{(1+n)_q}{q!}(-2)^p P_{n+p}^{(\alpha,\beta-p)}(x)L_{n+q}^{(\alpha-q)}(v)$$
$$\times y^{(\beta-p)}s^{(\alpha-q)}z^{(n+p)}t^{(n+q)}. \quad (1.1.26)$$

Finally substituting s = y = z = t = 1 in (1.1.26), we arrive at the proof of theorem.

We are able to determine two new classes of generating functions by utilizing the bilateral generating function for the set of Laguerre and Jacobi polynomials. These new classes are represented by theorem 1.1.1 and theorem 1.1.2. In the following section, we presented the applications to the newly investigated class of generating function, which are nothing more than specific examples of the conclusions shown above.

1.1.2 Application

If we put m = 0, we notice that $P_o^{(n,\beta)}(u) = 1$. Hence, from theorem 1.1.1, we deduce that

$$(1+w)^{\alpha}\exp(-wx)G(x+wx,w) = \sum_{n,p=0}^{\infty} a_n w^{n+p}\frac{(1+n)_p}{p!}L_{n+p}^{(\alpha-p)}(x). \tag{1.1.27}$$

1. If we put $a_n = 1$, in (1.1.27), we obtain

$$(1+w)^{\alpha}\exp(-wx)L_n^{(\alpha)}(x(1+w)) = \sum_{p=0}^{\infty} w^p \binom{n+p}{p} L_{n+p}^{(\alpha-p)}(x). \tag{1.1.28}$$

According to Das and Chatterjea's publication, this finding is identical to the one they previously got.

2. If we multiply both sides of (1.1.27) by r^n, we get

$$\begin{aligned}
&(1+w)^{\alpha}\exp(-wx)G(x+wx,wr) \\
&= \sum_{n,p=0}^{\infty} a_n w^{n+p}\frac{(1+n)_p}{p!} r^n L_{n+p}^{(\alpha-p)}(x), \\
&= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{n} a_{n-p}\binom{n}{p} r^{n-p} L_n^{(\alpha-p)}(x), \\
&= \sum_{n=0}^{\infty} w^n \sigma_n(x,r),
\end{aligned} \tag{1.1.29}$$

where

$$\sigma_n(x,r) = \sum_{s=0}^{\infty}\binom{n}{s} a_s r^s L_n^{(\alpha-n+s)}(x).$$

It is worth noting that this particular theorem is the first one in the paper written by Majumdar..

Through the application of the theorem 1.1.1, we were able to retrieve the well-known result of Majumdar, in addition to the results of Das and Chatterjea. This is what we mean when we say that we recovered the results.

1.2 LEGENDRE AND GEGENBAUER POLYNOMIALS

In order to finish the investigation of the subject, which is provided in the form of this thesis, we will now talk about the functions that generate Legendre and Gegenbauer polynomials. We make use of the book written by McBride to find novel generating relations that utilize modified Legendre polynomials. The ideas that are presented in the book are derived from the book called "Consent with Proofs."

It is generally accepted that Gegenbauer polynomials are solutions of a type of ordinary differential equations known as the hypergeometric equations.

A hypergeometric equation is defined as

$$p(x)G''(x) + t(x)G'(x) + \lambda G(x) = 0, \qquad (1).$$

Depending on the nature of p(x), different cases of (1) are given below:

(i) When p (x) is a constant, equation (1) becomes

$$G''(x) - 2\alpha x G'(x) + \lambda G(x) = 0.$$

When 〈=1, the Hermite polynomial is defined as

$$H_k(t) = (-1)^k \left(\frac{d^k}{dt^k} \{e^{-t^2}\} e^{t^2} \right), \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}^+, \quad \mathbb{Z}^+ = \{0, 1, 2, 3, \ldots, n\}.$$

(ii) If p(x) is a second degree polynomial, there are three different cases that can be obtained from equation (1). They are discussed under the following sub-labels as (a), (b) and (c):

(A) When p(x) has two different roots, equation (1) becomes the Jacobi equation:

$$(1 - x^2)G''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]G'(x) + \lambda G(x) = 0. \qquad (2)$$

When 〈 = ® (2) becomes the Gegenbauer polynomial under consideration.

When $\alpha = \beta = \pm\frac{1}{2},$ in (2), we have Chebyshev I and II polynomials.

When 〈 = ®=0, Legendre polynomial is obtained.

(B) When p(x) has double roots, equation (1) becomes

$$x^2 G''(x) + [(\alpha + 2)x + \beta]G'(x) + \lambda G(x) = 0. \quad (3)$$

Bessel polynomials are the solution of (3) when 〈 =□1 and ®=0

(a) Equation (1) takes the following form when p(x) has complex roots

$$(1 + x)^2 G''(x) + (2\beta x + \alpha)G'(x) + \lambda G(x) = 0. \quad (4)$$

(4) is generally referred to as Romanovski equation.

1.2.1 General Form of Gegenbauer Polynomial and the Generating Function

Gegenbauer polynomials are generally denoted by $C_n^{\lambda}(x)$. $C_n^{\lambda}(x)$ are particular solutions of (2) when 〈 = ®. Equation (2), when 〈 = ® also known as Gegenbauer differential equation, is given below:

$$(1 + x^2)y'' + (2\lambda - 1)xy' + n(n + 2\lambda)y = 0.$$

$C_n^{\lambda}(x)$ are sometimes called ultraspherical polynomials [4].

The generating function of the polynomial is given below:

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = (1 - 2xt + t^2)^{-\lambda}, \text{ for any } \lambda \in \mathbb{C}. \quad (5)$$

That is to say, $R^{-2\lambda} := (1 - 2xt + t^2)^{-\lambda}$ as the generating function. This means that $R := (1 - 2xt + t^2)^{\frac{1}{2}}$. Again, when $\lambda = \frac{1}{2},$ $C_n^{\lambda}(x)$ becomes the Legendre polynomial which is proportional to the ultraspherical polynomial $P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}$ and when $\alpha = \beta = \lambda - \frac{1}{2}$ the Jacobi polynomial $P_n^{(\alpha,\beta)}$ becomes ultraspherical polynomial.

Following is the generating function of the polynomial defined in (5):

$$C_0^{\lambda}(x) = (1 - 2xt + t^2)^{-\lambda},$$

$$C_0^{0}(x) = 1,$$

$$C_n^{0}(1) = \frac{2}{n},$$

$$C_n^{0}(x) = \lim_{\lambda \to 0} \frac{1}{\lambda} C_n^{\lambda}(x),$$

$$C_{2n}^{\lambda}(0) = \frac{(-1)^n}{n!} \frac{\Gamma(n+\lambda)}{\Gamma(n)}.$$

First few terms of the Gegenbauer polynomial are given below:

$$C_0^{\lambda}(x) = 1$$

$$C_1^{\lambda}(x) = 2\lambda x$$

$$C_2^{\lambda}(x) = 2\lambda(\lambda+1)x^2 - \lambda$$

$$C_3^{\lambda}(x) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)x^3 - 2\lambda(\lambda+1)x$$

$$C_4^{\lambda}(x) = \frac{2}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)x^4 - 2\lambda(\lambda+1)(\lambda+2)x^2 + \frac{1}{2}\lambda(\lambda+1)$$

1.2.2 Jacobi Identity on the Terms of Gegenbauer Polynomials

Definition (Lie algebra)

Let V be a finite dimensional vector space over the field K=(R,C) The vector space V is called a Lie algebra over $\supseteq$ if there is a rule of composition $(x+y) \to [X, Y] = XY - YX$ in V that satisfies the following axioms:

(i) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ for all $\alpha, \beta \in \mathbb{K}$ (linearity property);

(ii) $[X, Y] = -[Y, X]$ (antisymmetric property);

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \forall X, Y, Z \in V$.

The third axiom is called the Jacobi identity.

In this work, we examine the validity of this identity on the first three terms of the Gegenbauer polynomial:

$$C_0^{\lambda}(x) = 1,$$

$$C_1^{\lambda}(x) = 2\lambda x,$$

$$C_2^{\lambda}(x) = 2\lambda(\lambda + 1)x^2 - \lambda.$$

Let

$$X = C_0^{\lambda}(x),$$

$$Y = C_1^{\lambda}(x),$$

$$Z = C_2^{\lambda}(x).$$

The binary operation used here shall be composition of two functions. We define the Lie bracket [X,Y]=XY-YX. Using the composition as a binary operation,

$$[X, Y] = X \circ Y - Y \circ X.$$

We show that the Jacobi identity holds for the first three terms of the Gegenbauer polynomial under composition as binary operation:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0,$$

$$[XY - YX, Z] + [YX - ZY, X] + [ZX - XZ, Y] = 0,$$

$$[(XYZ - YXZ) - (ZXY - ZYX)] + [(YZX - ZYX) - (XYZ - XZY)]$$

$$+ [(ZXY - XZY) - (YZX - YXZ)] = 0.$$

Using composition of functions as the binary operation, we have

$$B = 4\lambda^3 + 2\lambda^2 - 8\lambda^4 - 8\lambda^3 + \lambda$$

$$= 8\lambda^4 + 8\lambda^3 - 2\lambda^2 - 4\lambda + 1,$$

$$C = [(Z \circ X \circ Y - X \circ Z \circ Y) - (Y \circ Z \circ X - Y \circ X \circ Z)],$$

$$Z \circ X \circ Y = (2\lambda(\lambda + 1)x^2 - \lambda) \circ (1) \circ (2\lambda x)$$

$$= 2\lambda^2 + \lambda,$$

$$X \circ Z \circ Y = (1) \circ (2\lambda(\lambda + 1)x^2 - \lambda) \circ (2\lambda x)$$

$$= 1,$$

$$Y \circ Z \circ X = (2\lambda x) \circ (2\lambda(\lambda + 1)x^2 - \lambda) \circ (1)$$

$$= 4\lambda^3 + 2\lambda^2,$$

$$Y \circ X \circ Z = (2\lambda x) \circ (1) \circ (2\lambda(\lambda + 1)x^2 - \lambda)$$

$$= 2\lambda,$$

$$C = 2\lambda^2 + \lambda - 1 - 4\lambda^3 - 2\lambda^2 + 2\lambda$$

$$= 3\lambda - 1 - 4\lambda^3.$$

Therefore, A+ B+ C=0 This implies that

$$1 - 4\lambda - 2\lambda^2 + 8\lambda^3 + 8\lambda^4 + 4\lambda^3 + 2\lambda^2 - 8\lambda^4 - 8\lambda^3 + \lambda + 3\lambda - 1 - 4\lambda^3 = 0.$$

Because of this, the Jacobi identity is satisfied on the first three terms of the Gegenbauer polynomial. According to this approach, any three terms of the Gegenbauer polynomial can be performed.

In mathematics, Gegenbauer polynomials or ultraspherical polynomials $C_n^v(x)$ are orthogonal polynomials on the interval [−1, 1] with respect to the weight function $(1-x^2)^{v-1/2}$. These polynomials are named after Leopold Gegenbauer.

The Gegenbauer polynomial $C_n^v(x)$ is a generalization of the Legendre polynomial $P_n(x)$ and is defined by the generating relation

$$\left(1-2xt+t^2\right)^{-v}=\sum_{n=0}^{\infty}C_n^v(x)t^n$$

(cf. [52] p.276,(1)). Legendre Polynomials Pn(x) for n = 0, 1, 2,... are defined as;

$$P_n(x)=(-1)^n{}_2F_1\begin{bmatrix}-n,n+1; & \\ & \frac{1+x}{2}\\ 1; & \end{bmatrix} \qquad (1.2.1)$$

We see that; these polynomials are particular solutions of Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

One may consult Rainville for more in details about Legendre Polynomials. On the other hand Gegenbauer polynomials $C_n^v(x)$, which are defined in their generalized form as;

$$C_n^v(x)=\frac{(2v)_n}{n!}{}_2F_1\begin{bmatrix}-n,2v+n; & \\ & \frac{1-x}{2}\\ v+\frac{1}{2}; & \end{bmatrix} \qquad (1.2.2)$$

are particular solutions of the Gegenbauer differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-(2v+1)x\frac{dy}{dx}+n(2v+n)y=0.$$

The following recurrence relations can be easily verified to show that Legendre and Gegenbauer polynomials satisfy the conditions presented. Please take note that the subscripts in the following relations are integers that are not negative.

$$(1-x^2)DP_n(x)=n\left[P_{n-1}(x)-xP_n(x)\right], \qquad (1.2.3)$$

$$(1-x^2)DP_n(x)=(n+1)\left[xP_n(x)-P_{n+1}(x)\right], \qquad (1.2.4)$$

$$(1-x^2)DC_n^v(x)=(2v-1+n)C_{n-1}^v(x)-nxC_n^v(x) \qquad (1.2.5)$$

and

$$(1-x^2)DC_n^v(x)=-(n+1)C_{n+1}^v(x)+(2v+n)xC_n^v(x), \qquad (1.2.6)$$

where D is the differential operator D = $\frac{d}{dx}$.

Conclusion

The generating functions for Laguerre, Legendre, and Gegenbauer polynomials provide a powerful tool for studying these important families of orthogonal polynomials. They offer a compact and elegant representation of the polynomials and can be used to derive various properties, such as recurrence relations, orthogonality conditions, and differential equations. By understanding the generating functions for these polynomials, we can gain deeper insights into their mathematical properties and applications in various fields, including physics, engineering, and numerical analysis.

References

[1] P. K. Banerji and F. B. F. Mohsen, On Generating Functions Involv- ing Laguerre and Modified Bessel Polynomials, Varahmihir Journal of Mathematical Sciences, Vol 2 No. 1, (2015), pp.191-196.

[2] P. K. Banerji and F. B. F. Mohsen, Bilateral generating relations for certain hypergeometric functions of more than one variable, South- East Asian J. math. math. sci. vol 1No. 1, (2018), pp.63-70.

[3] Bernardini and P. E. Ricci, Bell polynomials and differential equa-tions of Freud-type polynomials, Mathematical and Computer Modelling 36 (2019), pp.1115-1119.

[4] M. Bruschi and P. E. Ricci, Lucas and Chebyshev polynomials in several variables, Rendiconti di Matematica. Serie VI 13 (2020), no. 4, pp. 507–529.

[5] C. Cassisa and P. E. Ricci, Orthogonal invariants and the Bell poly- nomials, Rendiconti di Matema. Appl. Serie VII 20 (2020), pp. 293– 303.

[6] Di Cave and P. E. Ricci, On Bell polynomials and Fibonacci and Bernoulli numbers, Le Matematica 35 (2020), pp. 84–95.

[7] S. K. Chatterjea, On a pair of generating relations for some special functions from the view-point of Lie algebra, 27(1), (2021), pp. 11- 14.

[8] S. K. Chatterjea, Unification of a class of bilateral generating relations for certain special functions, Bull. Cal. Math. Soc., 67 (2020), pp. 115–127.

[9] S. K. Chatterjea, An extension of a class of bilateral generating functions for certain special functions, Bull. Inst. Math. Acad. Sinica 5(2) (2018).

[10] T. W. Chaundy, Expansions of Appeals Double Hypergeometric Functions, (11), Quart. J. Math. Oxford Series (2) 12(2019), pp. 112-128.